Golden Penney Ante

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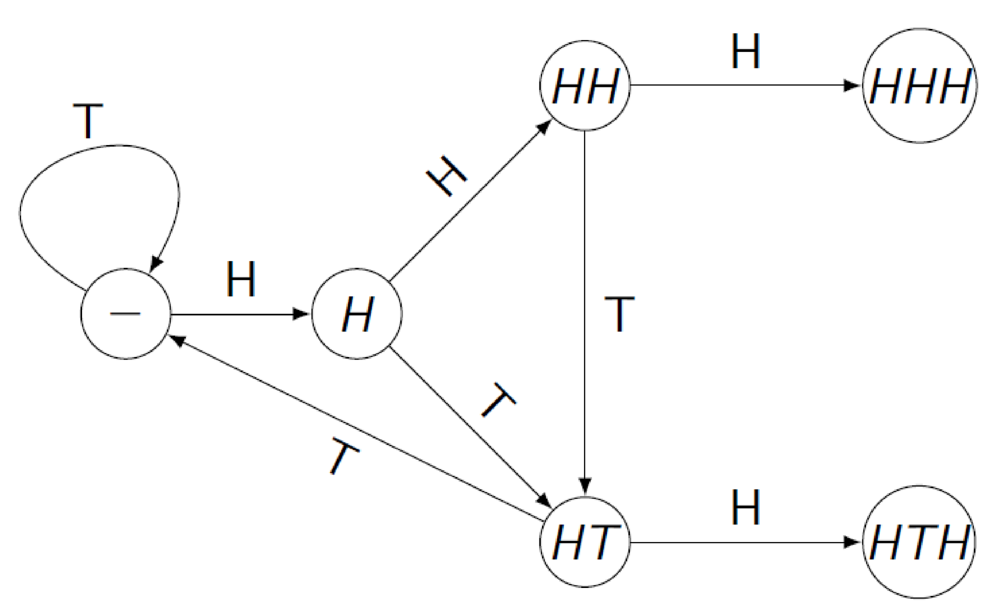
Abstract

This research is joint work with Mark Elmer of SUNY Oswego. In 1969, Walter Penney introduced his “Penney Ante” game, and in 1973 Craswell developed a memory state machine for computing probabilities in this game. We generalized Craswell’s results for arbitrary coin probabilities. In the analysis of the HHH vs HTH game, we noticed that the Golden Ratio played a role. Subsequently, we defined D(n,k) integers which are analogous to the binomial coefficients.

Pascal's Triangle displays the binomial coefficients in a triangle so that many interesting and important identities which relate these integers can be discovered. Our goal was to discover and to prove similar identities in the D(n,k) integers. We display the D(n,k) integers in a triangle similar to Pascal's triangle. Our preliminary analysis showed the existence of many analogous identities. We proved these identities through the techniques of mathematical induction and generating functions.

Craswell’s Memory State Machine

Player 1 HHH vs Player 2 HTH



Penney’s Results

Probabilities that Player 1 wins

Player 2

		HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
P	HHH		1/2	2/5	2/5	1/8	5/12	3/10	1/2
l	HHT	1/2		2/3	2/3	1/4	5/8	1/2	7/10
a	HTH	3/5	1/3		1/2	1/2	1/2	3/8	7/12
y	HTT	3/5	1/3	1/2		1/2	1/2	3/4	7/8
e	THH	7/8	3/4	1/2	1/2		1/2	1/3	3/5
r	THT	7/12	3/8	1/2	1/2	1/2		1/3	3/5
	TTH	7/10	1/2	5/8	1/4	2/3	2/3		1/2
1	TTT	1/2	3/10	5/12	1/8	2/5	2/5	1/2	

Generalization

We generalized Craswell’s results for arbitrary coin probability.
Let x be the probability heads, and so 1−x is the probability of tails (0 < x < 1). The probability that player 1 wins is $\frac{x}{1+x-x^2}$.

We wondered which coin probability will give a fair game for both players. So set $\frac{x}{1+x-x^2} = \frac{1}{2}$.

We obtain $x = \frac{\sqrt{5}-1}{2} \approx 0.618034$.

We see that x is the reciprocal of the Golden Ratio!

Combinatorial Analysis of HHH vs HTH

Let D(n,k) denote the number of winning player 1 sequences of the form t = (a_1a_2 ...a_rHHH) which have n number of tails and k number of heads in the block s = (a_1a_2...a_r) of t. The D(n,k) integers satisfy the following recurrence relationship:

D(n,0)= 1 for all n ≥ 0.
D(n,n)= 1 for all even n ≥ 0.
D(n,n)= 0 for all odd n > 0.
D(n,k)= D(n−1, k−1)+ D(n−1, k) for all n > k ≥ 1.

Theorem

For each real number x in (0,1),

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} D(n,k)x^k(1-x)^n = \frac{1}{x^2+x^3-x^4}.$$

We proved this using techniques from probability theory.

D(n,k) integers displayed in a triangle

1
1 0
1 1 1
1 2 2 0
1 3 4 2 1
1 4 7 6 3 0

Pascal’s Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Theorem

For each integer n ≥ 0,

$$D\binom{n}{k} = \sum_{i=0}^{k}(-1)^i\binom{n-i}{k-i}.$$

This provides a closed formula for the D(n,k) integers.

Theorem

Let n ≥ 0 be an integer and let $f(x) = \frac{1}{(1+x)(1-x)^{n+1}}$.

$$\text{Then } \frac{f^k(0)}{k!} = \sum_{i=0}^{k}(-1)^i\binom{n+k-i}{k-i} = D\binom{n+k}{k}.$$

Hence, for all k ≥ 0, $D\binom{n+k}{k}$ are the coefficients of the Maclaurin series expansion for f(x).

Theorem

Let b be any real number and let $f(x) = \frac{x}{1-bx-(b+1)x^2}$.

$$\text{Then } \frac{f^{n+1}(0)}{(n+1)!} = \sum_{k=0}^{n} D\binom{n}{k}b^{n-k}.$$

Hence, for all n ≥ 0, $\sum_{k=0}^{n} D\binom{n}{k}b^{n-k}$ are the coefficients of the Maclaurin series expansion for f(x).

References

Penney, W. (1969). Problem 95. Penney-Ante. *Journal of Recreational Mathematics 2*, p.241.
Craswell, K. (1973). An Interesting Penny Game. *The Two-Year College Mathematics Journal*, *4*(1), 18-25.